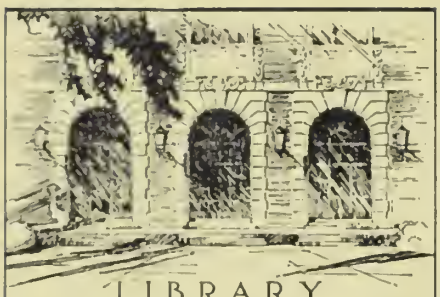

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A THEORETICAL ANALYSIS OF THE MODULATION CHARACTERISTICS OF AN ELECTRO-OPTIC LIGHT VALVE

by

DOUGLAS STUART SAND

JUL 29 1969

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DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN · URBANA, ILLINOIS

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A THEORETICAL ANALYSIS OF THE MODULATION CHARACTERISTICS OF AN ELECTRO-OPTIC LIGHT VALVE *

by

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January, 1969

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I. INTRODUCTION

The electro-optic effect is an interaction between high-frequency electromagnetic fields (such as light) and low-frequency electric fields (such as static fields or microwaves). The electro-optic effect is an example of a large class of phenomena (including optical second-harmonic generation, Stoke's radiation, and the stimulated Raman effect) generally called "nonlinear optics."[1], [2] Certain materials, such as the liquid nitrobenzene and the crystals quartz and KH_2PO_4 (KDP), exhibit a strong (measurable) electro-optic effect, [3], [4] as well as other nonlinear optical effects.

For analysis, the electro-optic effect is well approximated by expressing the optical dielectric properties of the material as a function of the local strength of the applied static field. Thus, for example, the local phase velocity (or index of refraction) becomes a function of the static field, so that the relative phase retardation of light passing through the material may be controlled by externally applied static or radio-frequency fields.[3]

In application the electro-optic effect is the basis for a variety of devices called "electro-optic light modulators." Perhaps the most familiar of these devices is the Kerr cell (or "light shutter"), used since the 1920's for high-speed photography. In recent years the electro-optic effect has been used in more sophisticated devices to produce modulation of phase, polarization, and amplitude of light,[3], [4] and to control the direction of light beam propagation.[4], [5], [6]

Most electro-optic light modulators are designed for time modulation of light, similar to modulation of a radio-frequency carrier. The "electro-optic light valve" (sometimes called an "Ardenne tube") is a spatial modulation

device; that is, the light valve imposes a modulation pattern across the surface area of a wavefront, analogous to the intensity-modulated wavefront of light which has passed through a photographic transparency. The light valve is potentially capable of reproducing a television image in the form of a "real-time photographic transparency." This potential has resulted in investigations in the use of the electro-optic light valve as part of a television projection system[7] and as the image source for real-time optical data processing.[8]

The configuration of a typical electro-optic light valve is shown in Figure 1. The light valve consists of a vacuum chamber containing an electron gun and a large flat thin electro-optic crystal, such as KDP. One face of the crystal is covered by a transparent conducting coating which is electrically grounded. The electron gun deposits a charge pattern on the other face of the crystal. The charge pattern induces a static electric field inside the crystal, so that light passing through is phase-modulated by the electro-optic effect. If the electron beam is scanned over the crystal face in a raster pattern and the beam current is controlled by a television video signal, the resulting charge pattern is a representation of a television picture. If the crystal is thin enough, the local charge density controls the strength of the local electric field, which in turn controls the local phase retardation of the emerging wavefront of light. Thus the emerging light is spatially modulated into a pattern resembling the original television image. A visible (intensity modulated) image may be obtained if the light is polarized before entering the crystal, and passes through an analyzer after emerging from the crystal (as in Figure 1).

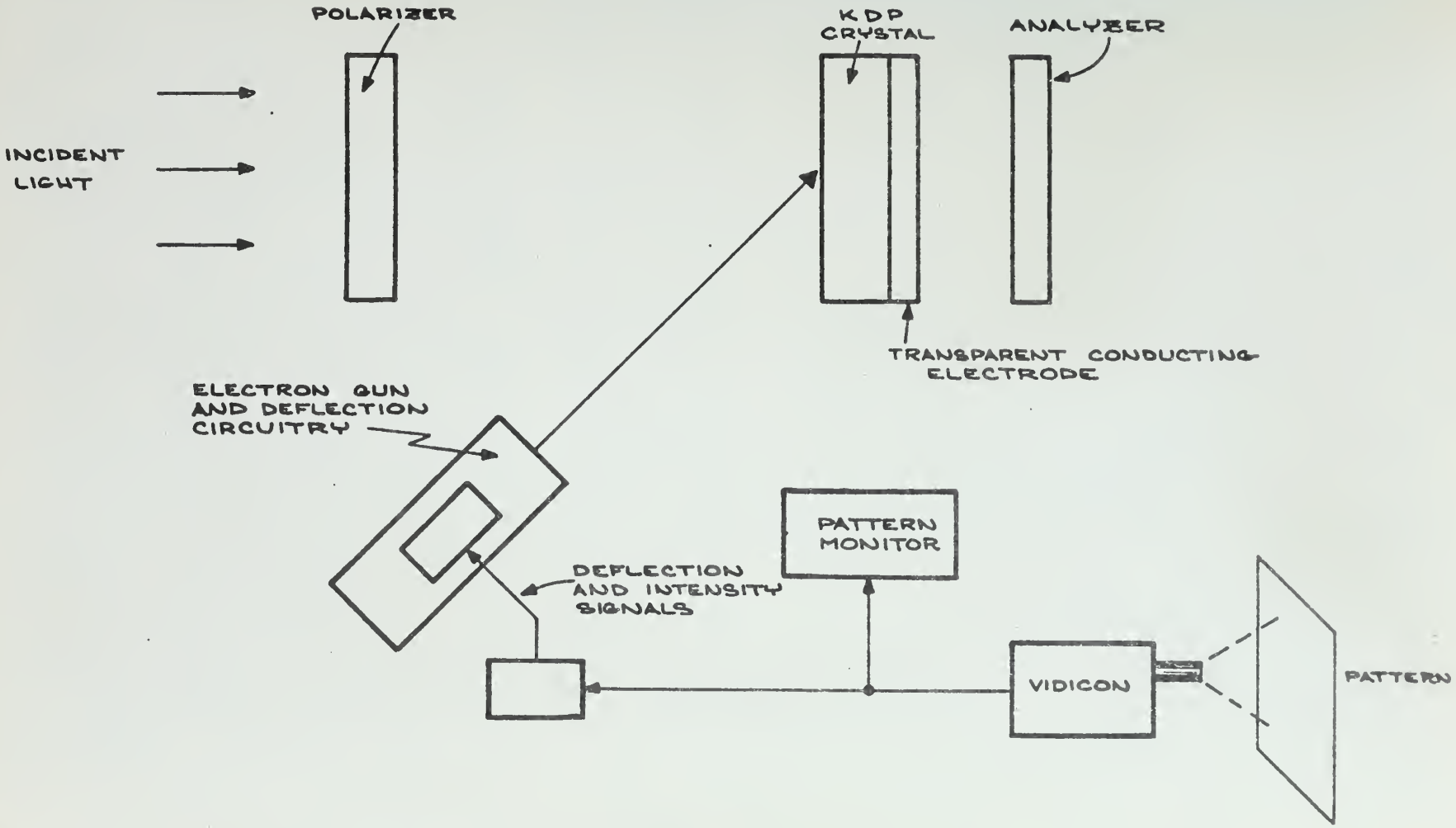

Figure 1. Electro-optic Light Modulator.

Various versions of the electro-optic light valve have been under development for about ten years.[9], [10], [11], [12] The physical construction and maintenance of these devices have presented many difficulties, but most have been solved by developing suitable technologies. A more serious problem has yet to be resolved: the modulation characteristics have been mediocre at best, so that - in particular - the maximum pattern resolution currently feasible is less than 25% of standard television resolution.

As will be seen, the pattern resolution is primarily determined by the thickness of the crystal relative to the size of its faces. This problem was recognized quite early,[9] and various attempts have been made to develop a theoretical analysis of the light valve modulation characteristics in order to provide a quantitative explanation of resolution. We now briefly consider two of these analyses.

The simplest (and perhaps earliest) analysis[9] considers a disk of diamater D located a distance T from a grounded conducting plane. The disk is charged to a potential V, and in the plane of the disk the surrounding potential falls to approximately V/2 at a diameter D + 0.44T. Assuming that the intensity (power) of the transmitted light is proportional to the potential in the plane of the spot, the half-power diameter of the light spot is equal to the V/2 diameter above. In general the electron beam spot size (~ D) is smaller than T, so the half-power diameter is approximately T/2. If we define the minimum resolvable separation of two light spots as approximately twice the half-power diameter, then resolution spacing is roughly T, the thickness of the crystal. Standard television resolution is 500 line-pairs, corresponding to 500 spots in a line, so that this resolution would require that the face of the crystal be at least 500 times the crystal thickness. Thus a 1" diameter crystal must be polished to less than 2 mils in thickness.

The second analysis[12], [13] is considerably more detailed and explicit than the first, but follows similar arguments. Again we have a disk at potential V located a distance T from the grounded conducting plane, but a quantitative expression is derived for the potential at all points in the plane of the disk. Further, it is assumed that the intensity of transmitted light is proportional to $\sin^2 [\alpha V(\rho) + \beta]$, where $V(\rho)$ is the disk-plane potential at a distance ρ from the spot center and where α and β are suitable constants (this $\sin^2$ dependence is found in typical electro-optic light modulators[3] which use an analyzer to convert polarization modulation to intensity modulation). The resulting expressions for the intensity distribution of the light spot are analyzed by a computer program to obtain resolution characteristics for a set of charge spot diameters and crystal thickness. Another approach[13] is used to obtain the normalized Fourier decomposition of the disk-surface potential V(x). This Fourier decomposition represents the spatial frequency response of the surface potential in terms of the charge pattern and can be used to obtain an equivalent "pass-band" for the device. This (normalized) Fourier decomposition V(N)/V(0) is given by

$$\frac{V(N)}{V(0)} = \frac{1 - e^{-2\pi NT}}{\pi \kappa NT} \left[(\kappa + 1) + (\kappa - 1)e^{-2\pi NT}\right]^{-1}$$

where N is the resolution in television lines (half-cycles) per millimeter, T is the crystal thickness in millimeters, and κ is the relative dielectric constant of the crystal. This spatial response analysis gives resolution characteristics similar to those of the light-spot analysis. The analysis concludes with a statement predicting that for a crystal of thickness T = 0.36 mm and a resolution of 500 television lines, the crystal face must be approximately 118 mm square. By linear interpolation we conclude that a 1" diameter crystal must have a thickness of less than 3 mils.

The quantitative results of these two analyses are similar: a 1" crystal must be polished to a thickness of 2-3 mils. Electro-optic crystals of useable quality (e.g., uniformity) are currently available in 1" - 2" sizes, but crystals of this size have not yet been polished to a thickness of less than about 10 mils. Thus tolerable resolution from an electro-optic light valve is in essence a technological problem, at least until crystals are available in the dimensions indicated by the preceding two analyses.

In considering these two resolution analyses, one finds they share two assumptions: First, the static fields are determined for an isotropic crystal; and second, the transmitted light intensity is considered a function of the surface potential only. As will be seen, the first assumption -- an isotropic crystal -- is reasonable since even though the electro-optic crystals are in general anisotropic, the static field equations and solutions have the same functional form in either an isotropic or an anisotropic dielectric. However, the resulting quantitative descriptions are somewhat different. The second assumption essentially implies that each element of the crystal face is considered as a local intensity modulator, with a uniform static field in the direction of light propagation determined by the local surface potential (as in a parallel plate capacitor). This local modulator is analyzed as a "standard" longitudinal (r_{63}) linear electro-optic polarization modulator.[3] That is, the crystal is normally uniaxial but becomes biaxial in the presence of a static field along the optic (x_3) axis. Light propagating through the crystal in the x_3 direction splits into two orthogonal components, each traveling at a different phase velocity controlled by the value of the static field. Leaving the crystal the two components recombine with a new polarization determined by the differential phase modulation and

thus by the value of the static field. As a first-order approximation for spatial as well as polarization modulation this analysis is only partially valid. Spatial modulation requires a local variation in the direction of propagation, so the light no longer travels along the optic axis. Since the two biaxial phase velocities are a function of direction of propagation, we might expect that pattern direction (in addition to static-field characteristics) will determine the spatial resolution of the light modulator. As will be seen in the first-order analysis that follows, the resolution characteristics are a function of direction, at least in theory.

The present work is an attempt to develop a more rigorous theoretical analysis of the modulation characteristics of the electro-optic light valve. In Section II the complete static field distribution is obtained for an anisotropic crystal, and for the spatial Fourier decomposition of an arbitrary charge pattern. This results in the anisotropic equivalent of the resolution characteristics V(N)/V(0) discussed previously. In Section III the static field solution is combined with the linear electro-optic dielectric tensor and Maxwell's equations, and a first-order solution of the resulting wave equation is obtained by a perturbation analysis. This first order solution is in the form of a far-field radiation pattern, giving a direct expression of the first-order optical modulation characteristics of the electro-optic light valve.

The static field solution and the first order optical solution are obtained by a generalized method for boundary conditions based on classical image sources (essentially an application of Green's theorem). These solutions and the method for their development are, to the best of the author's knowledge, original.

II. ANISOTROPIC STATIC FIELD SOLUTION

The static field equations for anisotropic media are very similar to those for ordinary isotropic media, since here the material is assumed to be homogeneous. However, the anisotropic parameters (the dielectric tensor ϵ) complicate the analysis somewhat. The physical configuration of the light valve is such that the general static field equations and boundary conditions can be simplified considerably, so that we will first define the configuration to be analyzed.

The configuration which describes the static field conditions appropriate for the light valve is shown in Figure 2. As discussed in Section I, the static field is established by a surface charge density located on one face of a flat thin crystal. This crystal has been oriented and cut so that the surfaces of the broad flat parallel faces are normal to one of the principal axes of the crystal. Let the coordinate system of the fields coincide with the principal axes of the crystal with, say, the x_3 - axis normal to the crystal faces. Let the x_3 - thickness of the crystal be T and choose the origin and direction of the x_3 axis so that the grounded conducting surface is on the right, at $x_3 = 0$, and the charged surface on the left, at $x_3 = -T$. Denote the surface charge distribution on the left by $\sigma(x_1, x_2)$ (this charge layer has been deposited by the electron beam). Since the size of the charge distributions to be considered is small compared with that of the face of the crystal, we can ignore edge effects and extend the crystal to infinity in the x_1, x_2 directions. Finally, we will refer to the left half-space, $x_3 < -T$, as Region 1 (free space), and to the crystal region, $-T \leq x_3 \leq 0$, as Region 2.

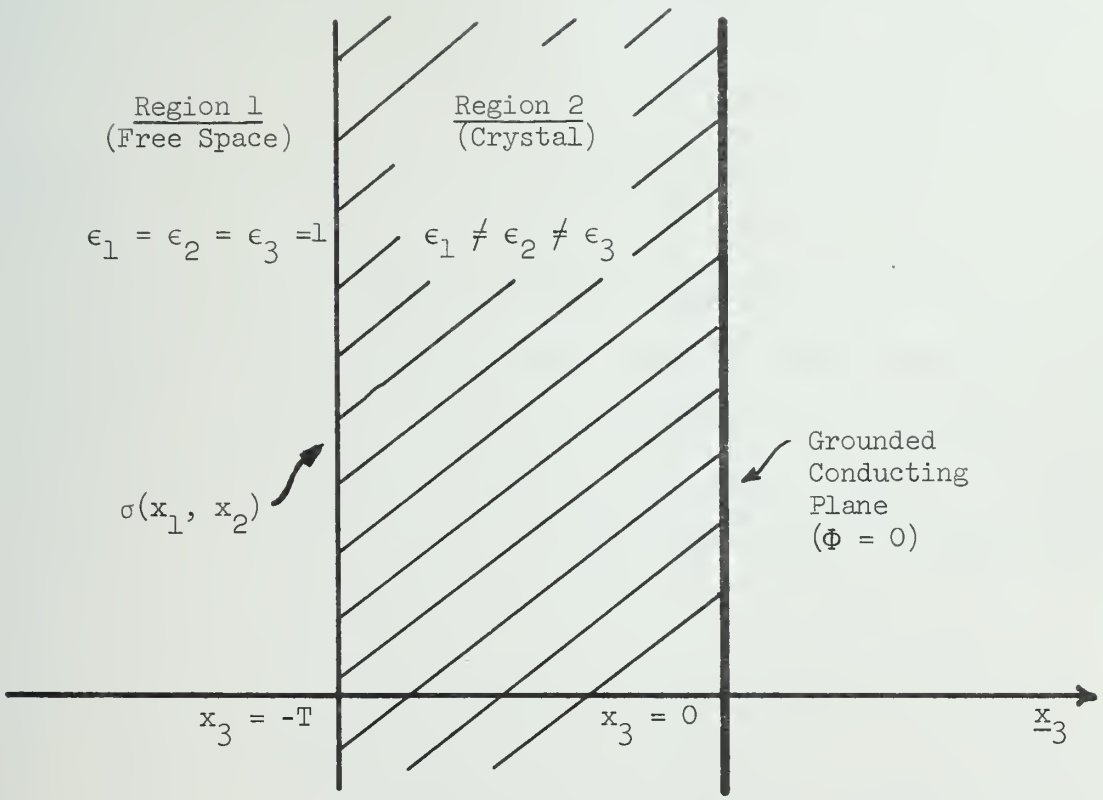

Figure 2. Crystal Configuration for Static Fields.

Maxwell's equations are independent of the properties of the material so that as usual the static field equations are

$$\nabla \times \underline{E} = 0 \tag{2.1}$$

$$\nabla \cdot \underline{D} = \rho \tag{2.2}$$

By (2.1) we may express the electric field $\underline{E}$ as the negative gradient of a scalar potential Φ:

$$\underline{E} = -\nabla\Phi \tag{2.3}$$

For a linear anisotropic material the electric material equation is in general a second-order tensor relation between $\underline{D}$ and $\underline{E}$. Here we assume the crystal is homogeneous so the elements of the relative dielectric tensor $\underline{\underline{\epsilon}}$ are constants. In the principal-axes coordinate system $\underline{\underline{\epsilon}}$ is diagonal, and the electric material equation reduces to

$$D_i = \epsilon_o \epsilon_i E_i, \quad i = 1, 2, 3 \tag{2.4}$$

Using (2.4), the divergence equation (2.2) becomes

$$\epsilon_1 \partial_1 E_1 + \epsilon_2 \partial_2 E_2 + \epsilon_3 \partial_3 E_3 = \rho/\epsilon_o \tag{2.5}$$

where the operator ∂_i denotes $\partial/\partial x_i$. In terms of the potential Φ the above equation (2.5) is

$$(\epsilon_1 \partial_1^2 + \epsilon_2 \partial_2^2 + \epsilon_3 \partial_3^2)\, \Phi = -\rho/\epsilon_o \tag{2.6}$$

This is the anisotropic Poisson's equation, similar to the isotropic Poisson's equation except for the coefficients ϵ_i. In fact, if we use the coordinate transformation, $x_i = u_i\sqrt{\epsilon_i}$, then (2.6) in the u_i - coordinate system is an isotropic Poisson's equation. By use of the inverse of the above transformation, any known isotropic solution for Poisson's equation can be converted into the equivalent anisotropic solution, if one is careful to observe that this coordinate transformation is not measure - preserving (that is, one must follow the general tensor rules for a coordinate transformation).

Let us denote the potential inside the crystal (Region 2) by Φ_R, and that in free space (Region 1) by Φ_L. The Φ_R must satisfy (2.6) with the ϵ_i determined by the particular crystal in use. However, Φ_L satisfies the free-space (isotropic) Poisson's equation, which is (2.6) with $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. In addition, Φ_L and Φ_R must satisfy the boundary conditions at the interfaces $x_3 = 0$ and $x_3 = -T$. The general anisotropic boundary conditions are rather complicated,[14] but in this case the surface normals coincide with the principal axes of the crystal. Then at the left-hand interface, continuity of the normal component of $\underline{D}$ requires

$$\partial_3\Phi_L - \epsilon_3\partial_3\Phi_R = \sigma/\epsilon_o \quad \text{at } x_3 = -T \tag{2.7}$$

where σ is the surface charge distribution described above. Continuity of the transverse component of $\underline{E}$ requires, as usual,

$$\Phi_L = \Phi_R \quad \text{at } x_3 = -T \tag{2.8}$$

Since the right-hand interface is grounded we must have

$$\Phi_R = 0 \quad \text{at } x_3 = 0 \tag{2.9}$$

Finally, since this grounded plane is conducting, the transverse component of $\underline{E}$ vanishes, requiring

$$\partial_1 \Phi_R = \partial_2 \Phi_R = 0 \quad \text{at } x_3 = 0 \tag{2.10}$$

These boundary conditions (2.7) through (2.10), together with the appropriate forms of Poisson's equation (2.6), are necessary and sufficient to completely define the static fields in terms of arbitrary source charge distributions σ and ρ. Actually, we will assume that ρ vanishes (since the resistivity of the crystal is very large), but it will prove convenient to retain ρ in (2.6) until later.

We will obtain the static field solutions by using Fourier transforms. This approach is advantageous for two reasons: first, to obtain the optical solutions we will use only the Fourier transform of the static field solutions; and second, in this particular case the Fourier transform solution is quite simple. We assume most functions of interest have three-dimension Fourier transforms, so that some scalar function $f(\underline{x})$ and its Fourier transform $f(\underline{k})$ are related by

$$f(\underline{k}) = \int_{-\infty}^{\infty} d^3x \, f(\underline{x}) \, e^{-j\underline{k} \cdot \underline{x}} \tag{2.11}$$

$$f(\underline{x}) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)3} \, f(\underline{k}) \, e^{j\underline{k} \cdot \underline{x}} \tag{2.12}$$

where

$$\underline{k} \cdot \underline{x} = k_1 x_1 + k_2 x_2 + k_3 x_3$$

By differentiating (2.12) with respect to x_i, it is easy to show that the functions

$$\partial_i f(\underline{x}),\ \nabla f(\underline{x}),\ \text{and}\ \nabla^2 f(\underline{x})$$

have as their respective Fourier transforms

$$jk_i f(\underline{k}),\ j\underline{k} f(\underline{k}),\ \text{and}\ -k^2 f(\underline{k})$$

where

$$k^2 = k_1^2 + k_2^2 + k_3^2$$

Thus the anisotropic Poisson's equation (2.6), which is in $\underline{x}$ - space representation, transforms into the $\underline{k}$ - space representation

$$(\epsilon_1 k_1^2 + \epsilon_2 k_2^2 + \epsilon_3 k_3^2)\ \Phi(\underline{k}) = \rho(\underline{k})/\epsilon_o \tag{2.13}$$

while the $\underline{k}$ - space isotropic Poisson's equation is simply

$$k^2\ \Phi(\underline{k}) = \rho(\underline{k})/\epsilon_o \tag{2.14}$$

The Fourier transforms of the boundary conditions (two - dimension transforms, since x_3 is fixed) will be considered later.

Let us for the moment consider the isotropic Poisson's equation in $\underline{k}$ - space, (2.14). For a unit point - charge located at $\underline{x} = \underline{a}$,

$$\rho(\underline{x}) = \delta(\underline{x} - \underline{a}) \equiv \delta(x_1 - a_1)\delta(x_2 - a_2)\delta(x_3 - a_3)$$

By (2.11) we evidently have

$$\rho(\underline{k}) = e^{-j\underline{k} \cdot \underline{a}}$$

so that $\Phi(\underline{k})$ is simply, from (2.14),

$$\Phi(\underline{k}) = e^{-j\underline{k} \cdot \underline{a}}/\epsilon_o k^2$$

It can easily be shown that the inverse transform of Φ above is given by

$$\Phi(\underline{x}) = 1/4\pi\epsilon_o r$$

where

$$r = |\underline{x} - \underline{a}|$$

This $\Phi(\underline{x})$ is obviously the familiar Coulomb potential of a unit point - charge. However, for our purposes we will transform $\Phi(\underline{k})$ back along only <u>one</u> direction, say x_3:

$$\Phi(k_1, k_2, x_3) = \int \frac{dk_3}{2\pi} \Phi(\underline{k}) e^{jk_3 x_3}$$

$$= e^{-j(k_1 a_1 + k_2 a_2)} \int \frac{dk_3}{2\pi\epsilon_o} \frac{e^{jk_3(x_3 - a_3)}}{k_1^2 + k_2^2 + k_3^2}$$

If we treat this integral as a contour integral, there are two poles in the complex k_3 - plane:

$$k_3 = \pm j\sqrt{k_1^2 + k_2^2} = \pm jk_r, \text{ say} \qquad (2.15)$$

This is an ordinary Fourier integral so that for convergence of a half - plane contour there are two cases of interest: for $x_3 > a_3$ we must use an upper half - plane contour ($\mathrm{Im}(k_3) > 0$), and for $x_3 < a_3$ we must use a lower half - plane contour. Then

$$\Phi(k_1, k_2, x_3) = \pm 2\pi j \Sigma \text{ (residues)}$$

$$= e^{-j(k_1 a_1 + k_2 a_2)} \frac{e^{-k_r|x_3 - a_3|}}{2k_r\epsilon_o}$$

This result is not remarkable in itself, but it suggests that a "mixed - space" representation (that is, (k_1, k_2, x_3) - space) might be particularly useful if the boundary conditions apply to x_3 only, since $\Phi(k_1, k_2, x_3)$ for a point charge is a simple exponential. We will make frequent use of this mixed - space representation in what follows.

Next we consider the method of images. Recall the elementary image - source distribution for a grounded conducting plane: the fields on

one side of the plane may be determined by replacing the conducting plane and opposite half - space with a negative mirror image of the charge distribution in the half - space of interest. In our case the conducting plane is defined by $x_3 = 0$. Using the isotropic Poisson's equation as an example, the potential $\Phi(\underline{x})$ then satisfies

$$-\nabla^2\Phi(\underline{x}) = \frac{1}{\epsilon_o}\left[\rho(x_1, x_2, x_3) - \rho(x_1, x_2, -x_3)\right] \tag{2.16}$$

where $\rho(\underline{x})$ is assumed to vanish for $x_3 > 0$. It is easily shown that $\Phi(\underline{x})$ determined by (2.16) completely satisfies the required boundary conditions at the grounded conducting plane ((2.9) and (2.10)) and satisfies the original Poisson's equation using only $\rho(\underline{x})$. By the uniqueness theorem this is the solution (in the half - space $x_3 < 0$ in this example). For use later, the $\underline{k}$ - space representation of (2.16) is

$$\epsilon_o k^2\Phi(\underline{k}) = \rho(k_1, k_2, k_3) - \rho(k_1, k_2, -k_3) \tag{2.17}$$

since from (2.11)

$$f(k_i) = -f(-k_i) \quad \text{if} \quad f(x_i) = -f(-x_i)$$

The method of images may be generalized to analyze a variety of physical configurations. As an example, consider an object - say, a chessman - seen through a thin lens: the image observed is a magnified copy of the original object. If the object and lens are replaced by an exact copy

of the object magnified to the proper image dimensions, the observed image is unchanged. Alternately, a hologram which has recorded the wavefront emerging from the lens will also reproduce the image seen by the observer. Both the hologram and the magnified copy of the object may be considered as image sources in the sense that each reproduces the observed image obtained from the original configuration. Note that in these two cases all conditions in the observation space ("real" space) are unchanged even though conditions in the image space (all space outside the observation space) are quite different. This example of generalized image sources suggests that image space conditions on the image source are ambiguous: we are free to define image space conditions in any self - consistent manner which does not disturb the original image in observation space. This statement may be rigorously demonstrated as a consequence of Green's integral theorems,[14], [15] but such a justification is beyond the scope of the present work. Finally, it is worth noting that the image source concept applies not only to true sources located outside the observation space, but also to the "reflected image" caused by true sources inside the observation space.

Since we are free to define image space conditions, let us separate the static field problem into two image source problems: first, a general solution for the potential Φ_L in terms of an image source ρ_L located outside Region 1, where Φ_L and ρ_L satisfy the free - space Poisson's equation everywhere; and second, a general solution for Φ_R in terms of an image source ρ_R located outside Region 2, where Φ_R and ρ_R everywhere satisfy the anisotropic Poisson's equation. The final solution will be determined by requiring that Φ_L and Φ_R satisfy the boundary conditions. By considering two separate full spaces, each homogeneous everywhere, we are allowed to use the $\underline{k}$ - space Poisson's equation as presented above (an alternate approach would be to use

the mixed - space Poisson's equation, which in this case would be a piece - wise linear ordinary differential equation).

Accordingly, from (2.14), the free - space solution is determined by

$$\Phi_L(\underline{k}) = \rho_L(\underline{k})/\epsilon_o k^2 \tag{2.18}$$

To evaluate the boundary conditions we must use the mixed - space representation for Φ_L, obtained from the ordinary Fourier integral

$$\Phi_L(k_1, k_2, x_3) = \int \frac{dk_3}{2\pi\epsilon_o} \frac{\rho_L(\underline{k})\, e^{jk_3x_3}}{k_3^2 + k_r^2} \tag{2.19}$$

where $k^2 = k_3^2 + k_r^2$ as in (2.15). We are interested in Region 1, $x_3 < -T$, and by definition the image source ρ_L must vanish everywhere in Region 1, so that singularities of $\rho_L(\underline{k})$ do not contribute to the contour integral of (2.19) evaluated for $x_3 < -T$. Then the only singularity of interest is the lower - half k_3 - plane pole $k_3 = -jk_r$. We then have, from (2.19),

$$\Phi_L(k_1, k_2, x_3) = \rho_L(k_1, k_2) \frac{e^{k_r x_3}}{2\epsilon_o k_r} \quad \text{for } x_3 < -T \tag{2.20}$$

where $\rho_L(k_1, k_2) \equiv \rho_L(k_1, k_2, k_3 = -jk_r)$.

To obtain the general solution for Φ_R in Region 2, we impose <u>two</u> conditions on the image source ρ_R. First, the image condition requires that ρ_R vanish in Region 2, for $-T < x_3 < 0$. For convenience we will use the form $\rho_R(x_3 + T)$, so that in $\underline{k}$ - space we have $\rho_R(\underline{k})e^{jk_3T}$ as the operational

form of the image source. Second, we will use the mirror image condition, as in (2.17), to satisfy the boundary condition at $x_3 = 0$. Then in $\underline{k}$ - space, from (2.13), we have

$$(\epsilon_1 k_1^2 + \epsilon_2 k_2^2 + \epsilon_3 k_3^2)\, \Phi_R = \rho_R(\underline{k}) e^{jk_3 T} + \rho_R(k_1, k_2, -k_3) e^{-jk_3 T} \tag{2.21}$$

Let us define

$$k_s = \sqrt{\epsilon_1 k_1^2 + \epsilon_2 k_2^2}/\sqrt{\epsilon_3} \tag{2.22}$$

Then because of the image condition the k_3 - plane singularities of interest are at $k_3 = \pm jk_s$. As before we evaluate an ordinary Fourier integral to obtain the mixed - space representation of Φ_R in Region 2:

$$\Phi_R(k_1, k_2, x_3) = - \frac{\rho_R(k_1, k_2) e^{-k_s T}}{\epsilon_o \epsilon_3 k_s} \sinh k_s x_3 \tag{2.23}$$

where $\rho_R(k_1, k_2) = \rho_R(k_1, k_2, k_3 = jk_s)$. It is easily shown that the Φ_R given by (2.23) satisfies the boundary conditions at $x_3 = 0$.

Now we apply the boundary conditions. It is clear that (2.7) and (2.8) transform directly into mixed - space representation, so that at $x_3 = -T$ we have

$$\partial_3 \Phi_L(k_1, k_2, -T) - \epsilon_3 \partial_3 \Phi_R(k_1, k_2, -T) = \sigma(k_1, k_2)/\epsilon_o \tag{2.24}$$

and

$$\Phi_L(k_1, k_2, x_3 = -T) = \Phi_R(k_1, k_2, x_3 = -T) \tag{2.25}$$

From (2.20) and (2.23) we obtain the required quantities for the mixed - space boundary conditions above, and, upon substitution, (2.24) and (2.25) reduce to

$$\rho_L e^{-k_r T} + 2\rho_R e^{-k_s T} \cosh k_s T = 2\sigma$$

and

$$\epsilon_3 k_s \rho_L e^{-k_r T} = 2k_r \rho_R e^{-k_s T} \sinh k_s T$$

From these two equations we obtain expressions for ρ_R and ρ_L which give the desired solutions:

$$\Phi_L(k_1, k_2, x_3) = \frac{\sigma(k_1, k_2)}{\epsilon_o} \frac{e^{k_r(x_3 + T)} \sinh k_s T}{k_r \sinh k_s T + \epsilon_3 k_s \cosh k_s T} \tag{2.26}$$

$$\Phi_R(k_1, k_2, x_3) = -\frac{\sigma(k_1, k_2)}{\epsilon_o} \frac{\sinh k_s x_3}{k_r \sinh k_s T + \epsilon_3 k_s \cosh k_s T} \tag{2.27}$$

Since the potential Φ expressed as (2.26) in Region 1 and (2.27) in Region 2 satisfies the appropriate boundary conditions and differential equations, by the uniqueness theorem this is <u>the</u> solution for the static fields in the light - valve configuration.

We will not attempt to evaluate the $\underline{x}$ - space representation of Φ except for the simple charge pattern consisting of a constant, say $\sigma(x_1, x_2) = A$. Then

$$\sigma(k_1, k_2) = (2\pi)^2 \delta(k_1) \delta(k_2) A$$

so that we may obtain $\Phi(\underline{x})$ by determining the limit of $\Phi(k_1, k_2, x_3)$ as k_1, k_2 vanish. If this is done, we find that

$$\Phi_R(\underline{x}) = -x_3 A/\epsilon_o\epsilon_3 \tag{2.28}$$

and

$$\Phi_L(\underline{x}) = TA/\epsilon_o\epsilon_3 \tag{2.29}$$

As might be expected, this potential is identical to that of an infinite parallel - plate capacitor filled with a material of relative dielectric constant ϵ_3.

To conclude this section we will comment on some of the immediate implications of the potential $\Phi_R(k_1, k_2, x_3)$ expressed by (2.27). Recall that the relative frequency response of a system is given by the Fourier transform of the system response to a unit impulse function. The unit impulse function for a two - dimension spatial pattern is $\delta(x_1)\ \delta(x_2)$, which is also $\sigma(x_1, x_2)$ for a unit point charge. For a unit point charge, $\sigma(k_1, k_2) = 1$, so $\Phi_R(k_1, k_2, x_3)$ with $\sigma = 1$ is the relative spatial frequency response of the potential function in any plane x_3 = constant (in Region 2). For example, at $x_3 = -T$ we have

$$\epsilon_o\Phi(k_1, k_2, x_3 = -T) = \frac{\sinh k_s T}{k_r \sinh k_s T + \epsilon_3 k_s \cosh k_s T}$$

This is a two - dimension spatial frequency response. The frequency response to a pattern of <u>lines</u> is given by $\Phi(k_1, k_2)$ evaluated along a line

in the k_1, k_2 plane. If, in general, $\epsilon_1 \neq \epsilon_2$ then the value of $k_s = \sqrt{\epsilon_1 k_1^2 + \epsilon_2 k_2^2}/\sqrt{\epsilon_3}$ (and, in turn, the relative line response) will depend on the direction of the k_1, k_2 line (which is normal to the direction of the x_1, x_2 lines). However, if we have a uniaxial crystal (such as KDP), then $\epsilon_1 = \epsilon_2$ and the frequency response to a line pattern is given by (setting $k_2 = 0$, say)

$$\Phi(k_1, 0, -T) = \frac{1}{\epsilon_o k_1} \frac{\sinh k_1 T'}{\sinh k_1 T' + \sqrt{\epsilon_1 \epsilon_3} \cosh k_1 T'}$$

where $T' = T\sqrt{\epsilon_1/\epsilon_3}$. If this $\Phi(k_1)$ is normalized (i.e., $\Phi(k_1)/\Phi(k_1 = 0)$) and the hyperbolic functions changed into exponentials, the resulting expression will be very similar to the function $V(N)/V(0)$ discussed in Section I, if we identify $k_1 T'$ in $\Phi(k_1)/\Phi(0)$ with the exponent πNT in $V(N)/V(0)$. The critical difference is that the <u>equivalent</u> crystal thickness $T' = T\sqrt{\epsilon_1/\epsilon_3}$ is a function of the ratio of the relative dielectric constants ϵ_1 and ϵ_3. For example, using published values[3] of ϵ_1 and ϵ_3 for some typical light-valve crystals we have

KH_2PO_4	(KDP)	$T' \sim 1.4T$
KD_2PO_4	(KD*P)	$T' \sim 1.1T$
$NH_4H_2PO_4$	(ADP)	$T' \sim 1.9T$

Thus, for a given crystal thickness, the normalized frequency response of the potential function should be best for KD*P, for which the effective thickness is nearly equal to the real thickness. However, this does not imply that of the three crystals KD*P is best for use in a light valve, since we have not considered the comparative effect of the electro-optic coefficients of each crystal.

III. FIRST - ORDER OPTICAL SOLUTION

Materials such as crystals which are electrically anisotropic for static and low - frequency fields are also anisotropic for optical - frequency fields. Thus the optical electric material equation is also a tensor relation, although the optical dielectric constants may differ considerably from the low - frequency dielectric constants, since dielectric properties are frequency - dependent. Further, it can be shown [2] that the electro-optic effect can be treated by introducing additional terms into the optical dielectric tensor. These electro-optic terms are small compared with the "natural" dielectric coefficients, so that they may be treated as a perturbation effect. Using this approach we will obtain a first - order perturbation solution for the propagation of light through the electro - optic crystal in the light valve.

The linear electric material equation is given by the tensor relation

$$\underline{D} = \epsilon_o \underline{\underline{\epsilon}} \cdot \underline{E} \tag{3.1}$$

where $\underline{\underline{\epsilon}}$ is the relative dielectric tensor. In the absence of the electro - optic effect the elements of $\underline{\underline{\epsilon}}$ are constants (the "natural" or unperturbed dielectric coefficients). To express the electro - optic effect additional terms (functions of the static field present) are added to the natural dielectric coefficients. The functional expression of these electro - optical terms depends on the symmetry class of the crystal being considered.[2] We will consider only the symmetry class $\bar{4}2m$ (which includes KDP and its isomorphs), for which $\underline{\underline{\epsilon}}$ has the following form in the principal -axes

coordinate system:

$$\underline{\underline{\epsilon}} = \begin{pmatrix} n_1^2 & a_3 & a_2 \\ a_3 & n_1^2 & a_1 \\ a_2 & a_1 & n_3^2 \end{pmatrix} \tag{3.2}$$

The constants n_1 and n_3 are respectively the ordinary and extraordinary indices of refraction ($n_1 = n_2$ so that $\bar{4}2m$ crystals are <u>uniaxial</u>). The quantities a_i are the linear electro - optic terms and are related to the static field $\underline{E}_s$ and the linear electro - optic constants r_{41}, r_{63} by[2]

$$\begin{aligned} a_1 &= -n_1^2 n_3^2 r_{41} E_{s1} \\ a_2 &= -n_1^2 n_3^2 r_{41} E_{s2} \\ a_3 &= -n_1^4 r_{63} E_{s3} \end{aligned} \tag{3.3}$$

In the light valve the static field is in general a function of position so that the a_i in (3.3) are not simple constants. In particular, the static field is determined by the potential Φ given by (2.27):

$$\Phi(k_1, k_2, x_3) = \beta(k_1, k_2) \sinh k_s x_3 \tag{3.4}$$

for $-T < x_3 < 0$ and where we have defined β as

$$\beta(k_1, k_2) = \frac{-\sigma(k_1, k_2)/\epsilon_o}{k_r \sinh k_s T + \epsilon_3 k_s \cosh k_s T} \tag{3.5}$$

By definition $\underline{E}_s = - \nabla\Phi$ so that in mixed - space representation (where the operator ∂_3 is unchanged but $\partial_1 \rightarrow jk_1$, $\partial_2 \rightarrow jk_2$) we have

$$\begin{aligned} a_1(k_1, k_2, x_3) &= jk_1 n_1^2 n_3^2 r_{41} \beta \sinh k_s x_3 \\ a_2(k_1, k_2, x_3) &= jk_2 n_1^2 n_3^2 r_{41} \beta \sinh k_s x_3 \\ a_3(k_1, k_2, x_3) &= k_s n_1^4 r_{63} \beta \cosh k_s x_3 \end{aligned} \tag{3.6}$$

As mentioned Section II, we will not need the $\underline{x}$ - space representation of the functions a_i above, since the analysis to follow uses the a_i in mixed - and $\underline{k}$ - space only.

It is convenient to separate the dielectric tensor $\underline{\underline{\epsilon}}$ into two terms, one containing the constant coefficients n_i and the other containing the functions a_i, so that (3.2) becomes

$$\underline{\underline{\epsilon}} = \underline{\underline{\epsilon}}_n + \underline{\underline{\chi}} \tag{3.7}$$

where

$$\underline{\underline{\epsilon}}_n \equiv \begin{pmatrix} n_1^2 & 0 & 0 \\ 0 & n_1^2 & 0 \\ 0 & 0 & n_3^2 \end{pmatrix} \tag{3.8}$$

and

$$\underline{\chi} \equiv \begin{pmatrix} 0 & a_3 & a_2 \\ a_3 & 0 & a_1 \\ a_2 & a_1 & 0 \end{pmatrix} \tag{3.9}$$

To obtain the appropriate electromagnetic field equations we must start with Maxwell's equations. We assume all field and source quantities are time - harmonic, so that, for example, the electric field vector $\underline{E}$ is given by

$$\underline{E}(\underline{x}, t) = \underline{E}(\underline{x})e^{-j\omega t} \tag{3.10}$$

Then Maxwell's equations, relating the usual field vectors $\underline{E}(\underline{x})$, $\underline{D}(\underline{x})$, $H(\underline{x})$, and $\underline{B}(\underline{x})$ to the source current density $\underline{J}(\underline{x})$ and the source charge density $\rho(\underline{x})$, are given by

$$\nabla \times \underline{H} = \underline{J} - j\omega\underline{D} \tag{3.11}$$

$$\nabla \times \underline{E} = j\omega\underline{B} \tag{3.12}$$

$$\nabla \cdot \underline{B} = 0 \tag{3.13}$$

$$\nabla \cdot \underline{D} = \rho \tag{3.14}$$

By combining the divergence of (3.11) with (3.14) we obtain another useful equation, the equation of source continuity:

$$\nabla \cdot \underline{J} = j\omega\rho \tag{3.15}$$

In addition to Maxwell's equations, we must use the material equations to relate $\underline{D}$ to $\underline{E}$ and $\underline{B}$ to $\underline{H}$. The electric material equation is given by (3.1), and the magnetic material equation for most crystals (in particular, KDP and its isomorphs) is simply the free - space permeability equation

$$\underline{B} = \mu_o \underline{H} \tag{3.16}$$

The above equations together with appropriate boundary conditions are sufficient to determine any of the field vectors. Further, if any one field is known, say $\underline{E}$, the other three are given directly by the first two of Maxwell's equations and the two material equations. This known field is the solution of a wave equation that relates the single vector to the sources. In our case the most convenient wave equation involves $\underline{E}$. One such wave equation for $\underline{E}$ is obtained by an appropriate combination of (3.11), the curl of (3.12), and the material equations (3.1) and (3.16):

$$\nabla \times \nabla \times \underline{E} - k_o^2 \underline{\underline{\epsilon}} \cdot \underline{E} = j\omega\mu_o \underline{J} \tag{3.17}$$

where $k_o^2 = \omega^2 \mu_o \epsilon_o$. However, this wave equation is difficult to analyze, because of the term $\nabla \times \nabla \times \underline{E}$. By use of the vector identity

$$\nabla \times \nabla \times \underline{E} \equiv \nabla(\nabla \cdot \underline{E}) - \nabla^2 \underline{E}$$

we obtain

$$\nabla^2 \underline{E} + k_o^2 \underline{\underline{\epsilon}} \cdot \underline{E} = -j\omega\mu_o \underline{J} + \nabla(\nabla \cdot \underline{E})$$

By an appropriate combination of (3.1), (3.14), (3.15) and the expression (3.7) for $\underline{\underline{\epsilon}}$ we have

$$n_1^2 \nabla \cdot \underline{E} = \frac{\nabla \cdot \underline{J}}{j\omega\epsilon_o} - \nabla \cdot \underline{\underline{\chi}} \cdot \underline{E} - (n_3^2 - n_1^2)\, \partial_3 E_3$$

When this expression for $\nabla \cdot \underline{E}$ is substituted into the wave equation we obtain

$$\nabla^2 \underline{E} + k_o^2 \underline{\underline{\epsilon}}_n \cdot \underline{E} + \left(\frac{n_3^2}{n_1^2} - 1\right) \nabla\partial_3 E_3 = -\left[\underline{\underline{I}} + \frac{1}{n_1^2 k_o^2} \nabla\nabla\right] \cdot \left[j\omega\mu_o \underline{J} + k_o^2 \underline{\underline{\chi}} \cdot \underline{E}\right] \qquad (3.18)$$

where $\underline{\underline{I}}$ is the identity tensor. For our purposes the wave equation (3.18) is more convenient than (3.17). Also, in free space $n_1 = n_2 = n_3 = 1$ and $\underline{\underline{\chi}}$ vanishes, so that the free - space wave equation corresponding to (3.18) is

$$\nabla^2 \underline{E} + k_o^2 \underline{E} = -j\omega\mu_o \left[\underline{\underline{I}} + \frac{1}{k_o^2} \nabla\nabla\right] \cdot \underline{J} = \underline{M}, \text{ say,} \qquad (3.19)$$

where the "modified" source current $\underline{M}$ is defined to be

$$\underline{M} = -j\omega\mu_o \left[\underline{\underline{I}} + \frac{1}{k_o^2} \nabla\nabla\right] \cdot \underline{J} \qquad (3.20)$$

Equation (3.19) represents an <u>uncoupled</u> partial differential equation. That is, (3.19) is three separate wave equations, one for each

component of $\underline{E}$. However, the crystal wave equation (3.18) is coupled because of the terms involving $\underline{\chi} \cdot \underline{E}$ and $\nabla\partial_3 E_3$. Further, since the coefficients of $\underline{\chi}$ are functions of position, (3.18) is a linear differential equation with non - constant coefficients, so that the solutions must be nonlinear functionals of the elements of $\underline{\chi}$. In the light valve this means that the visible picture is a nonlinear function of the applied charge pattern. As will be seen, however, the first - order perturbation solution is linear in form.

We have stated that the elements of the effective electro - optic tensor $\underline{\chi}$ are small relative to those of the dielectric tensor $\underline{\underline{\epsilon}}_n$. This is true if the magnitude of the static field is reasonable. For example, a typical light - valve crystal has $\epsilon_3 \sim 25$, $n_1 \sim n_3 \sim 1.5$, and $r_{63} \sim r_{41} \sim 10^{-11}$ m/v. Then for a typical crystal surface potential ~5kV and crystal thickness ~0.2mm, the magnitude of the static electric field is $\sim 10^6$ V/m, so that an element of $\underline{\chi}$ is $\sim 10^{-5}$ relative to an element of $\underline{\underline{\epsilon}}_n$. Thus $\underline{\chi}$ is indeed small compared with $\underline{\underline{\epsilon}}_n$, and a first order perturbation solution should give a reasonable approximation to the exact solution.

Recall that the perturbation approach[15] may be applied to a two - operator equation of the form

$$\underline{\underline{H}} \cdot \underline{E} = \underline{N} + \lambda \underline{\underline{G}} \cdot \underline{E}$$

which relates the solution vector $\underline{E}$ to the source vector $\underline{N}$. The operator $\underline{\underline{H}}$ represents the "unperturbed" system (for which, presumably, all solutions are known or can be obtained), and the operator $\underline{\underline{G}}$ represents the system

perturbation. The parameter λ is a small constant representing the "strength" of the perturbation (the perturbation is "turned on" as λ is increased from 0 to 1). The first order perturbation solution for $\lambda = 1$ is then given by

$$\underline{E} \sim \underline{E}_0 + \underline{E}_1$$

where $\underline{E}_0$ is the solution for the unperturbed system,

$$\underline{\underline{H}} \cdot \underline{E}_0 = \underline{N}$$

and $\underline{E}_1$ satisfies

$$\underline{\underline{H}} \cdot \underline{E}_1 = \underline{\underline{G}} \cdot \underline{E}_0$$

We will refer to $\underline{E}_0$ as the zero - order term and E_1 as the first - order term of the complete first - order solution. Note that the source $\underline{N}$ and the uniqueness theorem preclude the possibility that the unperturbed solution $\underline{E}_0$ is degenerate.

In the absence of the electro - optic effect (when the static field vanishes so that $\underline{\underline{\chi}} = 0$) the crystal wave equation (3.18) becomes

$$\nabla^2 \underline{E} + k_o^2 \underline{\underline{\epsilon}}_n \cdot \underline{E} + \left(\frac{n_3^2}{n_1^2} - 1\right) \nabla\partial_3 E_3$$

$$= -j\omega\mu_0 \left[\underline{\underline{I}} + \frac{1}{n_1^2 k_0^2} \nabla\nabla\right] \cdot \underline{J} \qquad (3.21)$$

$$= \underline{N}, \text{ say,}$$

where the modified crystal source current $\underline{N}$ is defined to be

$$\underline{N} = -j\omega\mu_o \left[\underline{\underline{I}} + \frac{1}{n_1^2 k_o^2} \nabla\nabla\right] \cdot \underline{J} \tag{3.22}$$

Evidently, the unperturbed system operator $\underline{\underline{H}}$ is the partial differential operator on the left of (3.21), so that the zero - order term $\underline{E}_o$ satisfies (3.21). Further, the perturbation operator $\underline{G}$ is clearly

$$-\left[k_o^2 \underline{\underline{I}} + \frac{1}{n_1^2} \nabla\nabla\right] \cdot \underline{\underline{\chi}}$$

so that the first - order term $\underline{E}_1$ satisfies (3.21) if the "zero - order source current" $j\omega\mu_o \underline{J}$ is replaced by the "first - order source current" $k_o^2 \underline{\underline{\chi}} \cdot \underline{E}_o$.

The uniaxial wave equation (3.21) only partially describes the unperturbed system, since it governs only the fields inside the crystal. All fields and sources outside the crystal must satisfy the free - space wave equation (3.19), and the dielectric discontinuity at the crystal surfaces introduces boundary conditions. The optical configuration of the crystal (see Figure 3) is similar to that for the static field: the geometry is identical but the right - hand interface is not a conducting plane (the conducting film -- such as vacuum - deposited gold -- is essentially transparent to light). We will use the same coordinate system as before and again assume the crystal extends to infinity in the $x_1 x_2$ plane. Regions 1 and 2 are defined as before, and Region 3 denotes the right half - space $x_3 > 0$. However, we use subscripts "L" and "R" to denote variables for Regions 1 and 3, respectively, and the subscript "C" for Region 2 variables.

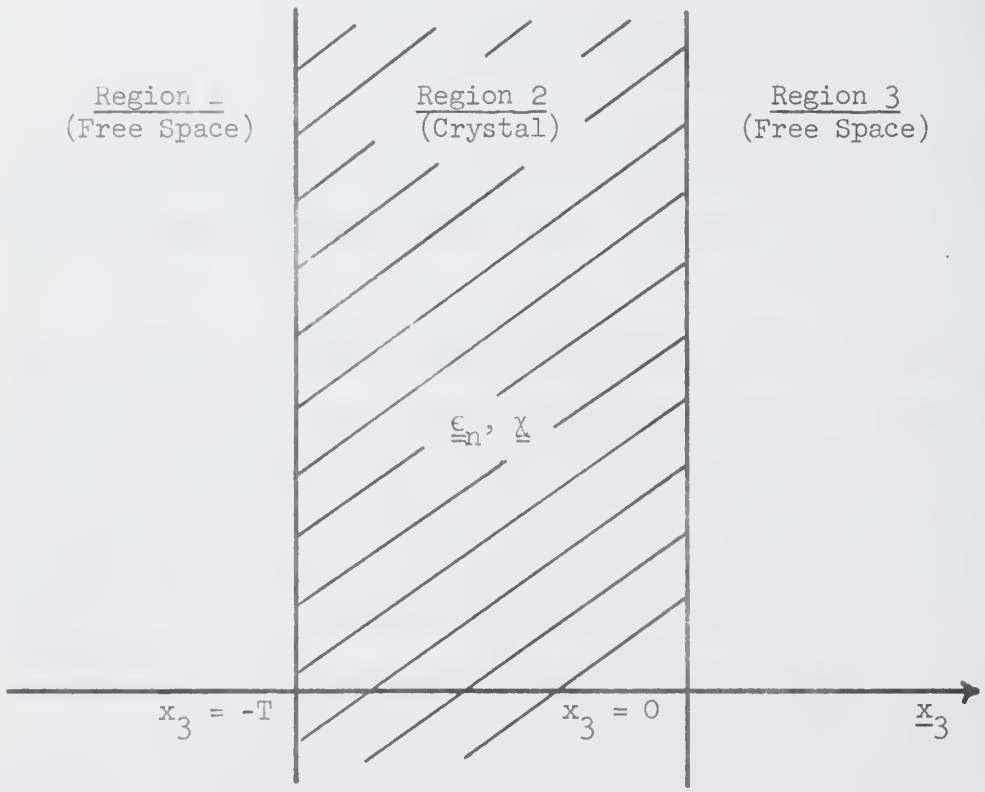

Figure 3. Crystal Configuration for Optical Fields.

Accordingly, the boundary conditions are as follows: continuity of the normal component of $\underline{D}$ and the transverse component of $\underline{E}$ requires

$$n_3^2 E_{C3} = E_{f3}$$
$$E_{C1} = E_{f1},\ E_{C2} = E_{f2} \tag{3.23}$$

at $x_3 = 0$ or $x_3 = -T$ (with the subscript "f" taken as "R" or "L"). Continuity of $\underline{H}$ at the interfaces requires

$$\nabla \times \underline{E}_C = \nabla \times \underline{E}_f \tag{3.24}$$

at each interface.

The Zero - Order Term. The zero - order term $\underline{E}_0$ can easily be obtained by conventional methods, since the "source" is an $\underline{x}_3$ - directed plane wave in Region 1, so we have normal incidence at the crystal interface $x_3 = -T$. Recall that for an isotropic dielectric sheet we assume that the solution vectors for a plane wave at normal incidence have no x_3 - components. Assuming then, that $\underline{E}_{0C}$ has no x_3 - component, we find that the uniaxial wave equation (3.21) reduces to

$$\nabla^2 \underline{E}_{0C} + n_1^2 k_o^2 \underline{E}_{0C} = 0$$

where $\underline{N} = 0$ since the source is confined to Region 1. This is identical to the isotropic wave equation so that the analysis and solution for the isotropic plane sheet are valid here. This solution[14] is indicated in

Figure 4 and is as follows. In Region 1 we have the incident plane wave $\underline{E}_{inc} = \underline{E}_{\alpha} e^{jk_o x_3}$, where $\underline{E}_{\alpha}$ is a constant vector and $\underline{E}_{\alpha} \cdot \underline{x}_3 = 0$. The reflected wave $\underline{E}_{refl}$ is of the form $e^{-jk_o x_3}$ and is related to the incident wave by a (reflection) coefficient Γ_L, where

$$\Gamma_L = e^{-j2k_o T} \frac{(n_1^2 - 1)\left[1 - e^{-j2n_1 k_o T}\right]}{(n_1 + 1)^2 e^{-j2n_1 k_o T} - (n_1 - 1)^2} \tag{3.25}$$

so that

$$\underline{E}_{OL} = \underline{E}_{\alpha} \left[e^{jk_o x_3} + \Gamma_L e^{-jk_o x_3}\right] \tag{3.26}$$

The transmitted wave $\underline{E}_{tran}$ is of the form $e^{jk_o x_3}$ in Region 3 and is determined by the (transmission) coefficient Γ_R,

$$\Gamma_R = \frac{4n_1 e^{-jk_o T}}{(n_1 + 1)^2 e^{-jn_1 k_o T} - (n_1 - 1)^2 e^{jn_1 k_o T}} \tag{3.27}$$

so that

$$\underline{E}_{OR} = \underline{E}_{\alpha} \Gamma_R e^{jk_o x_3} \tag{3.28}$$

In the crystal, Region 2, we have a positive - directed and a negative - directed plane wave, with wave number $n_1 k_o$. The solution for $\underline{E}_{OC}$ is then given by

$$\underline{E}_{OC} = \underline{E}_{\alpha} \Gamma_C \left[e^{jn_1 k_o x_3} + \eta e^{-jn_1 k_o x_3}\right] \tag{3.29}$$

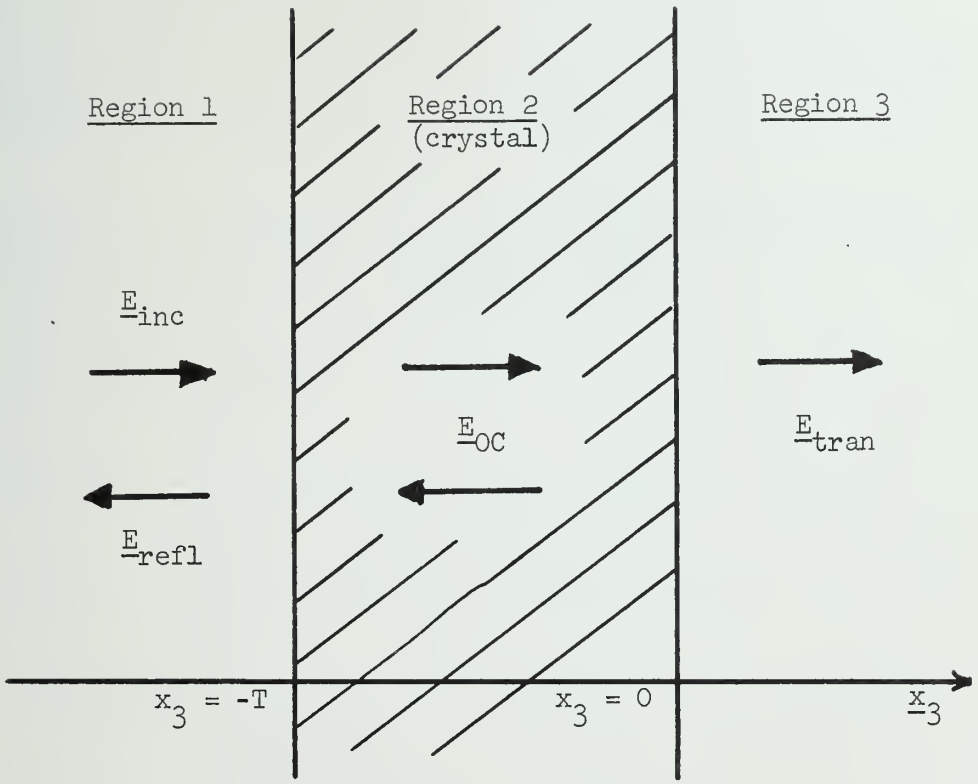

Figure 4. Zero - Order Optical Solution.

where

$$\eta \equiv \frac{n_1 - 1}{n_1 + 1} \tag{3.30}$$

and

$$\Gamma_C = \frac{2(n_1 + 1)\, e^{-jk_oT}}{(n_1 + 1)^2\, e^{-jn_1k_oT} - (n_1 - 1)^2\, e^{jn_1k_oT}} \tag{3.31}$$

It is not difficult to show that the solution for the zero - order term given above satisfies all the conditions required.

The First - Order Term. The solution for the zero - order term was easily obtained with an assumption suggested by the equivalent isotropic configuration. No such analogy is evident when the source is inside the crystal, as in the case of the first - order term. However, a solution can be found by a more general approach. For example, the method of images is appropriate here because of the form of the boundary conditions.

We will first obtain the general form of the solution in the free - space Regions 1 and 3. In $\underline{k}$ - space the wave equation (3.19) is

$$(k^2 - k_o^2)\, \underline{E}(\underline{k}) = - \underline{M}(\underline{k}) \tag{3.32}$$

If $\underline{M}$ is an image source, the mixed - space representation of $\underline{E}$ is determined by the singularity at $k_3 = \pm k_f$, where

$$k_f = (k_o^2 - k_r^2)^{1/2}, \quad k_r^2 = k_1^2 + k_2^2 \tag{3.33}$$

Since both k_3 singularities are on the path of integration, we invoke the radiation condition (which requires that a source emit waves) to select the desired singularity. Accordingly, $k_3 = -k_f$ represents waves propagating in the $-\underline{x}_3$ direction, so that the general solution for $\underline{E}_L$ in Region 1 is given by

$$\underline{E}_L(k_1, k_2, x_3) = \underline{M}_L \frac{e^{-jk_f x_3}}{2jk_f} \tag{3.34}$$

and in Region 3 we have

$$\underline{E}_R(k_1, k_2, x_3) = \underline{M}_R \frac{e^{jk_f x_3}}{2jk_f} \tag{3.35}$$

where

$$\begin{aligned} \underline{M}_L &= \underline{M}_L(k_1, k_2, k_3 = -k_f), \\ \underline{M}_R &= \underline{M}_R(k_1, k_2, k_3 = k_f) \end{aligned} \tag{3.36}$$

We are ultimately interested in an expression for $\underline{E}_R$, the first - order term for the transmitted wave.

The general forms of $\underline{E}_R$ and $\underline{E}_L$ and the boundary conditions can be used to obtain a set of conditions which the crystal field $\underline{E}_C$ must satisfy at the interfaces. Since there are no sources in Regions 1 and 3, the free - space fields must satisfy $\nabla \cdot \underline{E} = 0$ everywhere in the appropriate regions.

Thus, evaluation of $\nabla \cdot \underline{E} = 0$ (in mixed space) at the interfaces, followed by appropriate substitutions from the boundary conditions (3.23) yields

$$\begin{aligned} k_1 E_{C1} + k_2 E_{C2} - n_3^2 k_f E_{C3} &= 0 \text{ at } x_3 = -T \\ k_1 E_{C1} + k_2 E_{C2} + n_3^2 k_f E_{C3} &= 0 \text{ at } x_3 = 0 \end{aligned} \tag{3.37}$$

If we evaluate the free - space components of the boundary condition (3.24) and make appropriate substitutions from (3.23) we obtain four more independent relations for the interface values of $\underline{E}_C$, of which two contain E_{C1}:

$$\begin{aligned} jk_1(n_3^2 - 1)E_{C3} &= -(jk_f + \partial_3)E_{C1} \quad \text{at } x_3 = -T \\ jk_1(n_3^2 - 1)E_{C3} &= (jk_f - \partial_3)E_{C1} \quad \text{at } x_3 = 0 \end{aligned} \tag{3.38}$$

The other two equations are similar and relate E_{C2} and E_{C3}. An appropriate combination of these equations (3.38) yields

$$\begin{aligned} (k_f - j\partial_3)(k_2 E_{C1} - k_1 E_{C2}) &= 0 \quad \text{at } x_3 = -T \\ (k_f + j\partial_3)(k_2 E_{C1} - k_1 E_{C2}) &= 0 \quad \text{at } x_3 = 0 \end{aligned} \tag{3.39}$$

Another combination, of the equations in (3.38) and those in (3.37), results in

$$(n_3^2 k_o^2 - k_r^2 - jn_3^2 k_f \partial_3)(k_1 E_{C1} + k_2 E_{C2}) = 0 \quad \text{at } x_3 = -T$$
$$(n_3^2 k_o^2 - k_r^2 + jn_3^2 k_f \partial_3)(k_1 E_{C1} + k_2 E_{C2}) = 0 \quad \text{at } x_3 = 0 \tag{3.40}$$

To obtain the crystal field $\underline{E}_C$ we need the $\underline{k}$ - space uniaxial wave equation which, from (3.21), is

$$k^2 \underline{E}_C - k_o^2 \underline{\underline{\epsilon}}_n \cdot \underline{E}_C + \left(\frac{n_3^2}{n_1^2} - 1\right) \underline{k}\, k_3 E_{C3}$$
$$= -\underline{N} = \frac{j\omega\mu_o}{n_1^2 k_o^2} [n_1^2 k_o^2 \underline{J} - \underline{k}(\underline{k} \cdot \underline{J})] \tag{3.41}$$

It is possible to manipulate (3.41) such that three equations are obtained, one for each component of $\underline{E}_C$. With these equations one may easily obtain two useful expressions:

$$A_1 \equiv k_2 E_{C1} - k_1 E_{C2} = \frac{k_1 N_2 - k_2 N_1}{k_3^2 - k_c^2} \tag{3.42}$$

$$A_2 \equiv k_1 E_{C1} + k_2 E_{C2} = \frac{k_3 N_3}{k_3^2 - k_e^2} - \frac{\underline{k} \cdot \underline{N}}{k_3^2 - k_c^2} \tag{3.43}$$

where for convenience we have defined

$$k_c = (n_1^2 k_o^2 - k_r^2)^{1/2}$$
$$k_e = (n_1/n_3)(n_3^2 k_o^2 - k_r^2)^{1/2} \tag{3.44}$$

The definitions of A_1 and A_2 are appropriate because of the form of the

interface conditions (3.39) and (3.40) and because these expressions are useful in the analysis to follow.

The image source solution requires <u>two</u> image sources, one for each of the image regions $x_3 < -T$ and $x_3 > 0$. Until the boundary conditions are applied, the two image sources must be treated as independent functions and have the following characteristics: the left - hand image source $\underline{N}_L$ vanishes for $x_3 > -T$ and produces positive - directed waves in Region 2, and the right - hand image source $\underline{N}_R$ vanishes for $x_3 < 0$ and produces negative - directed waves in Region 2. These conditions on the image sources are required since Maxwell's equations are linear, and, in general, conditions at one boundary may be independent of those at the other.

For each of A_1 and A_2 we have $A_i = A_{Ni} + A_{Ii}$, where A_{Ni} is determined by the true source, and A_{Ii} by the image sources. If we define

$$k_c P_L = k_2 N_{L1} - k_1 N_{L2}$$

and

$$k_c P_R = k_2 N_{R1} - k_1 N_{R2}$$

then A_{I1} obtained from (3.42) has the mixed - space representation

$$2jA_{I1} = P_L e^{jk_c x_3} + P_R e^{-jk_c x_3} \tag{3.45}$$

Expressions for P_L and P_R in terms of the source function A_{N1} may be

obtained from the interface conditions (3.39), by substitution and evaluation of A_{I1} as given above. However, we only need A_{I1} at $x_3 = 0$, for which we have, from (3.40) and (3.45)

$$P_L + P_R = -2j \frac{2k_c C_1(-T) + C_1(0)[(k_c - k_f)e^{jk_cT} + (k_c + k_f)e^{-jk_cT}]}{(k_c + k_f)^2 e^{-jk_cT} - (k_c - k_f)^2 e^{jk_cT}} \tag{3.46}$$

where

$$C_1(-T) \equiv (k_f - j\partial_3) A_{N1} \quad \text{at } x_3 = -T$$

and

$$C_1(0) \equiv (k_f + j\partial_3) A_{N1} \quad \text{at } x_3 = 0$$

The function $A_1 = A_{N1} + A_{I1}$ is, then, at $x_3 = 0$:

$$A_1 = A_{N1} + \frac{1}{2j}(P_L + P_R) \tag{3.47}$$

where $(P_L + P_R)$ is obtained as above.

To determine the appropriate expression for A_{N2} from (3.43), we note that by definition in (3.41), the quantity $\underline{k} \cdot \underline{N}$ reduces to

$$\underline{k} \cdot \underline{N} = \frac{-j\omega\mu_o}{n_1^2 k_o^2} (n_1^2 k_o^2 - k^2)(\underline{k} \cdot \underline{J})$$

so that an alternate expression for (3.43) is

$$A_2 = \frac{k_3 N_3}{k_3^2 - k_e^2} - \frac{j\omega\mu_o}{n_1^2 k_o^2}\,\underline{k} \cdot \underline{J} \tag{3.48}$$

For the image function A_{I2}, we require that the image sources vanish in Region 2, so the term in $\underline{k} \cdot \underline{J}$ does not affect the mixed - space representation of A_{I2} in Region 2. Accordingly, we have

$$-2jA_{I2} = N_{L3} e^{jk_e x_3} - N_{R3} e^{-jk_e x_3}$$

so that evaluation of the interface conditions (3.40) yields

$$N_{L3} - N_{R3} \tag{3.49}$$
$$= 2j\,\frac{2n_1^2 k_f C_2(-T) + C_2(0)[(k_e + n_1^2 k_f)e^{-jk_eT} - (k_e - n_1^2 k_f)e^{jk_eT}]}{(k_e + n_1^2 k_f)^2 e^{-jk_eT} - (k_e - n_1^2 k_f)^2 e^{jk_eT}}$$

where

$$C_2(-T) \equiv \frac{1}{k_e}(k_e^2 - jn_1^2 k_f \partial_3) A_{N2} \quad \text{at } x_3 = -T$$

and

$$C_2(0) \equiv \frac{1}{k_e}(k_e^2 + jn_1^2 k_f \partial_3) A_{N2} \quad \text{at } x_3 = 0$$

Then we have $A_2 = A_{N2} + A_{I2}$ evaluated at $x_3 = 0$:

$$A_2 = A_{N2} - \frac{1}{2j}(N_{L3} - N_{R3}) \tag{3.50}$$

where $(N_{L3} - N_{R3})$ is given above.

To evaluate the A_{Ni} we need the first - order current source, determined by the dot - product of the electro - optic tensor (3.9) and the zero - order crystal field (3.29). With the following definitions

$$\begin{aligned} g_1 &\equiv \Gamma_c \beta n_1^2 n_3^2 r_{41} \\ g_3 &\equiv \Gamma_c \beta n_1^4 r_{63} \end{aligned} \tag{3.51}$$

and

$$\begin{aligned} F_1 &\equiv E_{\alpha 2} \\ F_2 &\equiv E_{\alpha 1} \\ F_3 &\equiv k_1 F_1 + k_2 F_2 \end{aligned} \tag{3.52}$$

and using (3.6) we obtain the x_1 - component

$$(\underline{\chi} \cdot \underline{E}_{oc})_1 = F_1 g_3 k_s \cosh k_s x_3 (e^{jn_1 k_o x_3} + \eta e^{-jn_1 k_o x_3}) \tag{3.53}$$

The x_2 - component is given by a similar expression, with F_2 replacing F_1. For the x_3 - component we have

$$(\underline{\underline{\chi}} \cdot \underline{E}_{OC})_3 = jF_3 g_1 \sinh k_s x_3 (e^{jn_1 k_o x_3} + \eta e^{-jn_1 k_o x_3}) \qquad (3.54)$$

Since the electro - optic effect is confined to Region 2 and vanishes elsewhere, the $\underline{k}$ - space representation of $\underline{\underline{\chi}} \cdot \underline{E}_{OC}$ is obtained by a x_3 Fourier integral, integrating from $x_3 = -T$ to $x_3 = 0$.

Such a "truncated" Fourier integral is equivalent to a convolution integral of the form

$$\int_{-T}^{0} dx f(x) e^{-jkx} = \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \frac{e^{jT(k - k')} - 1}{j(k - k')} f(k')$$

where $f(k')$ is the ordinary Fourier integral (2.11) in one dimension only. We will denote the above convolution integral by

$$R(k - k') * f(k')$$

where

$$R(k) = (e^{jkT} - 1)/jk$$

With the definition

$$\underline{L}(\underline{k}) \equiv \int_{-\infty}^{\infty} dx_3 e^{-jk_3 x_3} \underline{\underline{\chi}}(k_1, k_2, x_3) \cdot \underline{E}_{OC}(x_3)$$

the first - order source current becomes

$$j\omega\mu_o \underline{J}(\underline{k}) = k_o^2 R(k_3 - k'_3) * \underline{L}(\underline{k}') \tag{3.55}$$

where $\underline{k}' \equiv (k_1, k_2, k'_3)$. Then from (3.53) we have

$$L_1(\underline{k}) = \pi F_1 g_3 k_s [\delta(k_3 - k_\alpha^*) + \delta(k_3 - k_\alpha) + \eta\delta(k_3 + k_\alpha) + \eta\delta(k_3 + k_\alpha^*)] \tag{3.56}$$

where

$$k_\alpha \equiv n_1 k_o + jk_s$$

A similar expression is obtained for L_2, and from (3.54) we have

$$L_3(\underline{k}) = j\pi F_3 g_1 [\delta(k_3 - k_\alpha^*) - \delta(k_3 - k_\alpha) + \eta\delta(k_3 + k_\alpha) - \eta\delta(k_3 + k_\alpha^*)] \tag{3.57}$$

We have defined the first - order source current as the convolution (3.55) because the delta functions in $\underline{L}$ reduce the convolution to a simple sum over the appropriate values of k'_3. If a mixed - space representation is to be obtained, the k'_3 convolution may be deferred until after the k_3 transform of $R(k_3 - k'_3)$ is obtained.

We now evaluate the mixed - space expression for A_1. From (3.41), (3.42), and (3.55) we have

$$A_{N1}(\underline{k}) = -k_o^2 \frac{R(k_3 - k'_3)}{k_3^2 - k_c^2} * [k_1 L_2(\underline{k}') - k_2 L_1(\underline{k}')] \tag{3.58}$$

If we define

$$G_c(\underline{k}, k_3') = -R(k_3 - k_3')/(k_3^2 - k_c^2)$$

and evaluate the mixed - space representation at the interfaces we obtain

$$G_c(\underline{k}', x_3 = 0) = \frac{1 - e^{jT(k_c - k_3')}}{2k_c(k_c - k_3')}$$

$$G_c(\underline{k}', x_3 = -T) = \frac{e^{-jk_3'T} - e^{jk_cT}}{2k_c(k_c + k_3')}$$

and the derivatives reduce to

$$\partial_3 G_c(\underline{k}', x_3 = 0) = jk_c G_c(\underline{k}', x_3 = 0)$$

$$\partial_3 G_c(\underline{k}', x_3 = -T) = -jk_c G_c(\underline{k}', x_3 = -T)$$

The relation between the interface value of $\partial_3 G_c$ and that of G_c is independent of k_3', so that after convolution, the same relation holds between A_{N1} and $\partial_3 A_{N1}$ at the interfaces. This result reduces the expressions (3.46) and (3.47) to

$$A_1(k_1, k_2, x_3 = 0) = 2k_c \frac{(k_c - k_f)A_{N1}(-T) + (k_c + k_f)A_{N1}(0)e^{-jk_cT}}{(k_c + k_f)^2 e^{-jk_cT} - (k_c - k_f)^2 e^{jk_cT}} \qquad (3.59)$$

where

$$A_{N1}(-T) \equiv A_{N1}(k_1, k_2, x_3 = -T)$$

with a similar expression for $A_{N1}(0)$, at $x_3 = 0$.

The exact expression for A_1 at $x_3 = 0$ is then obtained by substituting into (3.59) the interface values of G_c for those of A_{N1} and evaluating the k_3' convolution as indicated by the definition of G_c and (3.58). The result is the following expression:

$$A_1(k_1, k_2, x_3 = 0) =$$

$$\frac{g_3 k_s k_o^2 (k_1 F_2 - k_2 F_1)}{[4\gamma n_1^2 k_o^2 + (\gamma-1)k_r^2][(k_c+k_f)^2 e^{-jk_cT} - (k_c-k_f)^2 e^{jk_cT}]}$$

$$\times \{[k_c(\gamma-1)(1+\eta) + n_1 k_o(\gamma+1)(\eta-1)](k_c+k_f)e^{-jk_cT}$$

$$- [k_c(\gamma-1)(1+\eta) - n_1 k_o(\gamma+1)(\eta-1)](k_c-k_f)e^{jk_cT}$$

$$- 2k_c \cosh k_s T[k_f(\gamma-1)(e^{-jn_1k_oT} + \eta e^{jn_1k_oT})$$

$$- n_1 k_o(\gamma+1)(e^{-jn_1k_oT} - \eta e^{jn_1k_oT})]$$

$$- 2j(\gamma/k_s)\sinh k_s T[2n_1 k_o k_f(e^{-jn_1k_oT} - \eta e^{jn_1k_oT})$$

$$+ (2n_1^2 k_o^2 + k_r^2(\gamma-1))(e^{-jn_1k_oT} + \eta e^{jn_1k_oT})]\}$$

where $\gamma = \epsilon_1/\epsilon_3$. As can be seen, this exact expression is quite complicated. However, a reasonable approximation results in considerable simplification. We are discussing an optical solution, so that the free-space wave-number $k_o \equiv 2\pi/\lambda_o$ is very large. Thus for light of $\lambda_o \sim 0.6\mu m$, we have $k_o \sim 10^7 m^{-1}$. This value is to be compared with typical values of k_1 and k_2. From (3.51) and (3.5) we find that the above expression for $A_1(0)$ contains the factor $\sigma(k_1,k_2)$ -- the Fourier composition of the crystal charge pattern -- so that the meaningful range of k_1 and k_2 is determined by the maximum frequency content of $\sigma(k_1,k_2)$. For standard television we have a maximum resolution of 500 lines (half-cycles) in any direction, which, for a typical crystal size of 25mm square, gives $\sim 10^4 m^{-1}$ for a maximum value of k_1 or k_2. Then for, say, $k_f^2 = k_o^2 - k_1^2 - k_2^2$, the approximation $k_f \sim k_o$ is very good for all values of k_1,k_2 of interest. Also, from (3.44), we have $k_c \sim n_1 k_o$. These approximations reduce the expression for $A_1(0)$ to only one term of magnitude k_o relative to the remaining terms. Retaining only this k_o term, we obtain

$$A_1(k_1,k_2,x_3 = 0) = k_2 E_{c1} - k_1 E_{c2} \cong$$

$$(jg_3 k_o \sinh k_s T)\frac{k_2 F_1 - k_1 F_2}{n_1+1}\ \frac{1 + \eta^2 e^{j2n_1 k_o T}}{1 - \eta^2 e^{j2n_1 k_o T}} \qquad (3.60)$$

where, as before, $\eta = (n_1-1)/(n_1+1)$.

To obtain an expression for A_2 at $x_3 = 0$, we begin with (3.48), using the first - order source current (3.55):

$$A_{N2} = \frac{k_3 N_3}{k_3^2 - k_e^2} - \frac{1}{n_1^2} R * [\underline{k} \cdot \underline{L}(k')]$$

By substitution for N_3, from (3.41), this becomes

$$A_{N2} = R^* \frac{n_3^2 k_o^2 (k_1 L_1 + k_2 L_2) - n_1^2 k_r^2 (\underline{k} \cdot \underline{L})}{n_3^2 (k_3^2 - k_e^2)} \qquad (3.61)$$

When this expression is evaluated, the term obtained from the $(k_1 L_1 + k_2 L_2)$ component of (3.61) will be of order k_o relative to that obtained from the $\underline{k} \cdot \underline{L}$ component. This is reasonable since the $\underline{k} \cdot \underline{L}$ term, when evaluated, obtains a factor of order k_o (from $k_3 L_3$) while the term for $(k_1 L_1 + k_2 L_2)$ already possesses the factor k_o^2, as in (3.61). Since the approximation for A_1, (3.60), contains only the terms of highest order in k_o, we may then approximate A_{N2} by

$$A_{N2} \sim k_o^2 R^* \frac{k_1 L_1 + k_2 L_2}{k_3^2 - k_e^2} \qquad (3.62)$$

Note that this approximation implies that only r_{63} modulation is retained in this first-order evaluation, since the factor r_{41} occurs only in the term $\underline{k} \cdot \underline{L}$. The form of (3.62) above is similar to that for A_{N1} in (3.58), and the analysis to follow parallels that for A_1. If we define a $G_e(\underline{k}, k_3')$ as in the definition for G_c above (but with k_e substituted for k_c), the interface expressions for G_e and $\partial_3 G_e$ have the same form as those for G_c and $\partial_3 G_c$, so that equations (3.49) and (3.50) defining A_2 reduce to

$$A_2(k_1,k_2,x_3 = 0) \sim$$

$$2n_1^2k_f \frac{(k_e + n_1^2k_f)e^{-jk_eT}A_{N2}(0) - (k_e - n_1^2k_f)A_{N2}(-T)}{(k_e + n_1^2k_f)^2e^{-jk_eT} - (k_e - n_1^2k_f)^2e^{jk_eT}} \qquad (3.63)$$

where $A_{N2}(-T) = A_{N2}(k_1,k_2,x_3 = -T)$ with a similar definition for $A_{N2}(0)$, at $x_3 = 0$. As before, we substitute G_e for A_{N2} in (3.63) and evaluate the convolution integral. The result is

$$A_2(k_1,k_2,x_3 = 0) \cong$$

$$\frac{g_3n_1^2k_fk_s(k_o^2 - \nu k_r^2)(k_1F_1 + k_2F_2)}{[4\gamma n_1^2k_o^2 + (\gamma-\nu)k_r^2][(n_1^2k_f + k_e)^2e^{-jk_eT} - (n_1^2k_f - k_e)^2e^{jk_eT}]}$$

$$\times \{[(1+\eta)(\gamma-\nu) - (1-\eta)(\gamma+\nu)(n_1k_o/k_e)](k_e+n_1^2k_f)e^{-jk_eT}$$

$$+ [(1+\eta)(\gamma-\nu) + (1-\eta)(\gamma+\nu)(n_1k_o/k_e)](k_e-n_1^2k_f)e^{jk_eT}$$

$$- 2n_1\cosh k_sT[n_1k_f(\gamma-\nu)(e^{-n_1k_oT} + \eta e^{jn_1k_oT})]$$

$$-k_o(\gamma+\nu)(e^{-jn_1k_oT} - \eta e^{jn_1k_oT})]$$

$$- 2j(\gamma/k_s)\sinh k_sT[2n_1^3k_ok_f(e^{-jn_1k_oT} - \eta e^{jn_1k_oT})$$

$$+ (2n_1^2k_o^2 + k_r^2(\gamma-\nu))(e^{-jn_1k_oT} + \eta e^{jn_1k_oT})]\}$$

where $\nu = n_1^2/n_3^2$. As before, we use the appropriate approximations, $k_e \sim n_1 k_o$ and $k_f \sim k_o$, to reduce the above expression to only one term of relative order k_o. Discarding all but this highest order term, the result is

$$A_2(k_1,k_2,x_3 = 0) = k_1 E_{c1} + k_2 E_{c2} \cong$$

$$(-jg_3 k_o \sinh k_s T) \frac{k_1 F_1 + k_2 F_2}{n_1 + 1} \qquad (3.64)$$

Finally, by comparison of (3.37) and (3.64) it is evident that to order k_o we may neglect the terms involving E_{c3} at $x_3 = 0$, so that our solution involves only E_{c1} and E_{c2}.

The approximations for E_{c1} and E_{c2} at $x_3 = 0$ are easily obtained by appropriate combinations of (3.60) and (3.64). For example, we have

$$E_{c1}(k_1,k_2,x_3 = 0) = (k_2 A_1 + k_1 A_2)/k_r^2$$

$$\cong \frac{-jg_3 k_o \sinh k_s T}{(n_1+1)(1 - \eta^2 e^{2jn_1 k_o T})}$$

$$\times \{\frac{2F_2 k_1 k_2}{k_r^2} + F_1[\frac{k_1^2 - k_2^2}{k_r^2} - \eta^2 e^{2jn_1 k_o T}]\} \qquad (3.65)$$

For E_{c1} the expression is similar to (3.65) differing only by the interchange of F_1 and F_2 as well as that of k_1 and k_2.

We can obtain a more compact expression by a change in coordinate representation, to k_1, k_2 polar coordinates. Let

$$\theta_k = \tan^{-1} k_2/k_1 \tag{3.66}$$

so that

$$k_1 = k_r \cos\theta_k \quad \text{and} \quad k_2 = k_r \sin\theta_k$$

Then the factor in braces in (3.65) reduces to

$$F_1 \cos 2\theta_k + F_2 \sin 2\theta_k - F_1 \eta^2 e^{j2n_1 k_o T}$$

Further, recall (3.52) relating F_1 and F_2 to $\underline{E}_\alpha$, which by (3.26) defines the magnitude and polarization of the incident plane wave. Let θ_α define the angle of polarization, so that

$$\begin{aligned} F_1 &= E_{\alpha 2} = E_\alpha \sin\theta_\alpha \\ F_2 &= E_{\alpha 1} = E_\alpha \cos\theta_\alpha \end{aligned} \tag{3.67}$$

(Note that elliptical polarization can be described by a complex-valued angle.[16]) This definition for θ_α, together with certain obvious triginometric identities, allow a further reduction, so that (3.65) becomes

$$E_{c1} \cong \frac{-jg_3 k_o \sinh k_s T}{(n_1 + 1)(1 - \eta^2 e^{2jn_1 k_o T})}$$

$$\times E_\alpha[\sin(2\theta_k + \theta_\alpha) - \eta^2 e^{2jn_1 k_o T} \sin\theta_\alpha] \qquad (3.68)$$

Now, from (3.4) and (3.51) we have

$$g_3 \sinh k_s T = n_1^4 r_{63} \Gamma_c \Phi(-T)$$

where

$$\Phi(-T) = \Phi(k_1, k_2, x_3 = -T)$$

This reduces (3.68) to

$$E_{c1}(k_1, k_2, x_3 = 0) \cong \frac{-jk_o n_1^4 r_{63} \Phi(-T) \Gamma_c E_\alpha}{(n_1 + 1)(1 - \eta^2 e^{2jn_1 k_o T})}$$

$$\times [\sin(2\theta_k + \theta_\alpha) - \eta^2 e^{2jn_1 k_o T} \sin \theta_\alpha] \qquad (3.69)$$

By a similar procedure we obtain

$$E_{c2}(k_1, k_2, x_3 = 0) \cong \frac{jk_o n_1^4 r_{63} \Phi(-T) \Gamma_c E_\alpha}{(n_1 + 1)(1 - \eta^2 e^{2jn_1 k_o T})}$$

$$\times [\cos(2\theta_k + \theta_\alpha) + \eta^2 e^{2jn_1 k_o T} \cos \theta_\alpha] \qquad (3.70)$$

Finally, from (3.37) it is evident that the approximation for E_{c_3} at $x_3 = 0$ is not of order k_o and so may be neglected.

We have now obtained the approximation for $\underline{E}_C$ at $x_3 = 0$, which, together with the boundary conditions (3.23) and the expression (3.35) for $\underline{E}_R$, determine the first - order term of the transmitted wave in Region 3. The approximation for the first - order solution in Region 3 is given by the sum of $\underline{E}_R$ as in (3.35) and the zero - order transmitted wave $\underline{E}_{OR}$ from (3.28). However, since the resulting expression would be rather complicated (as in (3.70)) we will consider a special case.

If the light - valve is used to obtain visible pictures (intensity modulation) then the incident light is polarized, and the transmitted light passed through an analyzer oriented perpendicular to the polarization of the incident light. In particular, if the incident light is polarized in the x_1 - direction, say, we would be interested in the amplitude (or intensity) of the E_2 component of the transmitted wave relative to that of the E_1 component. If we obtain the ratio of E_{C2}, given by (3.70), to $(\underline{E}_{OR})$ at $x_3 = 0$, given by (3.28), we should have a good estimate of the amplitude modulation characteristics for this case. For E_1 - polarization we have $\theta_\alpha = 0$ and $F_1 = E_{\alpha 2} = 0$, so from (3.27) through (3.31),

$$\underline{E}_{OR}(x_3=0) = \underline{x}_1 E_\alpha \Gamma_R = \underline{x}_1 E_\alpha \Gamma_C \frac{2n_1}{n_1 + 1} \tag{3.71}$$

Then, from (3.70) and (3.71),

$$(E_{c2}/E_{OR1})\big|_{x_3=0} \cong \frac{jn_1^3 r_{63} k_o \Phi(-T)}{2(1 - \eta^2 e^{2jn_1 k_o T})} \times [\cos 2\theta_k + \eta^2 e^{2jn_1 k_o T}] \tag{3.72}$$

where $\Phi(-T) = \Phi(k_1, k_2, x_3 = -T)$ as given by (2.27). Now the far - field diffraction pattern can be shown[16] to be related to the Fourier decomposition of the source pattern, which in our case ($k_f \sim k_o$) is simply the ratio given above (the ratio is used since it is independent of the amplitude and phase of the incident light). If we recall the discussion in Section II concerning the static field resolution at $x_3 = -T$, the function $\Phi(-T)$ determines the pattern resolution characteristics of the static field. From the ratio given above, $\Phi(-T)$ is also quite important in determining the pattern resolution characteristics of the transmitted light. However, the presence of the factor

$$\cos 2\theta_k + \eta^2 e^{2jn_1 k_o T}$$

indicates that the pattern resolution is a function of <u>direction</u> in the pattern, and that the response should be minimum along the direction determined by $\theta_k = \pi/4$. This directional dependence is not unreasonable, since we have observed that the quantity $k_1 E_{C1} + k_2 E_{C2}$ is associated with the wave - number k_e, which in turn is related to the extraordinary wave number $n_3 k_o$, and similarily, $k_2 E_{C1} - k_1 E_{C2}$ is related to the ordinary wave number $n_1 k_o$. The two components then travel at different

velocities inside the crystal, and reach the interface $x_3 = 0$ out of phase. The result is an amplitude which depends on the values of k_1^2 and k_2^2. Thus the pattern resolution characteristics of the light valve are, to this first - order approximation, closely related to those of the static - field surface potential, but resolution is non-uniform as a function of k_1, k_2 direction.

Finally, we compare the result (3.72) to that obtained by considering each element of the crystal to be a local longitudinal polarization modulator. In this approximation we have[3]

$$E_2 \sim \gamma E_{OR1} \sin(\pi\Phi/\Phi_\lambda)$$

where γ is a factor related to Γ_c, $\Phi = \Phi(-T)$ is the crystal surface potential, and Φ_λ is the full-wave retardation voltage which for r_{63} modulation is given by

$$\Phi_\lambda = \lambda/n_1^3 r_{63} = 2\pi/n_1^3 r_{63} k_o$$

Substitution for Φ_λ yields

$$E_2 \sim \gamma E_{OR1} \sin \tfrac{1}{2} n_1^3 r_{63} k_o \Phi(-T)$$

Then, using the Taylor expansion

$$\sin x = x - \frac{x^3}{3!} + \ldots .$$

we have, for small Φ,

$$E_2/E_{OR1} \sim \frac{\gamma}{2} n_1 {}^3 r_{63}{}^k{}_o \Phi(-T)$$

Clearly, the first-order term above is almost identical to the form of (3.72), except for the absence of the θ_k dependence. It would be of interest to obtain an experimental evaluation of the resolution as a function of θ_k. If the predicted θ_k dependence is not observed, then the first-order approximation is not adequate, so that the perturbation series must then include higher-order terms.

SUMMARY

Our aim in the present work has been to develop a theoretical analysis of the modulation characteristics of an electro - optic light valve. The development has been based on Maxwell's equations and a physical configuration which is representative of that of a typical light valve. We have attempted to use a minimal number of assumptions and to justify approximations by order - of - magnitude estimates. The fundamental assumption is that the linear electro - optic effect may be expressed by an appropriate optical dielectric tensor.[2]

In Section II we obtained the exact expression for the static field present in the anisotropic light - valve crystal, in terms of the Fourier composition of an arbitrary surface charge distribution. In Section III we used the linear electro - optic tensor to express the crystal wave equation as a function of the static field. A first - order perturbation treatment was used to obtain an approximation for the optical fields transmitted through the crystal. This approximation was then used to obtain an expression for the resolution characteristics of an intensity - modulation light valve. The development for the solutions of both the static fields and the optical fields was based on a generalized method of images.

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